

SIMULTANEOUS DETERMINATION OF TWO UNKNOWN THERMAL COEFFICIENTS THROUGH AN INVERSE ONE-PHASE LAMÉ-CLAPEYRON (STEFAN) PROBLEM WITH AN OVERSPECIFIED CONDITION ON THE FIXED FACE†

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Abstract—Formulas are obtained for the simultaneous determination of two of the four coefficients, k (thermal conductivity), l (latent heat of fusion), c (specific heat), ρ (mass density), of a material occupying a semi-infinite medium. This determination is obtained through an inverse one-phase Lamé-Clapeyron (Stefan) problem with an overspecified condition on the fixed face of the phase change material. To solve this problem, we assume that the coefficients $h_0, \sigma, \theta_0 > 0$ are known from experiments (where h_0 characterizes the heat flux through the fixed face, σ characterizes the moving boundary and θ_0 is the temperature on the fixed face). Denoting the temperature by θ , the results we obtain concerning the associated moving boundary problem are the following:

- (i) When one of the triples $\{\theta, k, l\}$, $\{\theta, k, \rho\}$ is to be found, the corresponding moving boundary problem always has a solution of the Lamé-Clapeyron-Neumann type.
- (ii) If one of the triples $\{\theta, k, c\}$, $\{\theta, l, c\}$, $\{\theta, l, \rho\}$, and $\{\theta, c, \rho\}$ has to be determined, the above property is satisfied if and only if a complementary condition for the data is verified.

Formulas are also obtained for the simultaneous determination of other physical coefficients and the inequality $\xi^2 < Ste/2$ (Ste : Stefan number) for the coefficient ξ of the free boundary $s(t) = 2a\xi t^{1/2}$ of the Lamé-Clapeyron solution of the one-phase Stefan problem without unknown coefficients.

NOMENCLATURE

- c , specific heat;
- f , error function;
- k , thermal conductivity;
- l , latent heat of fusion;
- h_0 , coefficient defined by equation (2e);
- Ste , Stefan number, cl_0/l ;
- s , position of phase change location;
- t , time variable;
- x , spatial variable.

Greek symbols

- α , thermal diffusivity, $k/\rho c$ ($= a^2$);
- ρ , mass density;
- σ , coefficient defined by equation (1);
- θ , temperature;
- θ_0 , temperature on the fixed face, $x = 0$;
- ξ , dimensionless parameter, σ/a .

1. INTRODUCTION

SUPPOSE that two of the four coefficients, k (thermal conductivity), l (latent heat of fusion), c (specific heat), ρ

(mass density) of a phase (e.g. liquid) of some given material are known. If, by means of a change of phase experiment (fusion of the material at its melting temperature) we are able to measure the quantities $h_0 > 0$, $\sigma > 0$ and $\theta_0 > 0$, then we will be able to find the formulas for the simultaneous determination of the unknown coefficients.

Consider the inverse one-phase Lamé-Clapeyron problem (or the inverse one-phase Stefan problem with constant thermal coefficients) [7, 18, 35, 45, 54, 60] with an overspecified condition on the fixed face $x = 0$ [53, 54]. This overspecified condition consists of the specification of the heat flux through the fixed face of the material undergoing the phase change process. Other boundary value problems for the 1-dim. heat equation with an overspecified condition on a part of the boundary have been analyzed [5, 8, 9, 11-17, 22, 23, 29-31, 42]. (See also the references listed in ref. [54].)

In ref. [59], it was shown that the solution of the inverse conduction problem is characterized by a discontinuous dependence on data. Other references dealing with inverse problems are refs. [3, 38, 47]; those on the identification of parameters are refs. [19-21, 24, 34, 49], and those on improperly posed problems in partial differential equations are refs. [36, 37, 43, 50, 58].

If we suppose that the melting temperature is zero and the moving boundary is given by

$$s(t) = 2\sigma t^{1/2}, \text{ with } \sigma > 0, \quad (1)$$

our problem is reduced to finding the temperature

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$\theta = \theta(x, t)$ of the liquid phase, defined for $0 < x < s(t)$ and $t > 0$, and two of the four coefficients k, l, c, ρ such that they satisfy the following conditions:

$$\frac{\partial \theta}{\partial t} = a^2 \frac{\partial^2 \theta}{\partial x^2}, \quad 0 < x < s(t), \quad t > 0, \tag{2a}$$

$$\theta[s(t), t] = 0, \quad t > 0, \tag{2b}$$

$$-k \frac{\partial \theta}{\partial x}[s(t), t] = \rho l s'(t), \quad t > 0, \tag{2c}$$

$$\theta(0, t) = \theta_0, \quad t > 0, \tag{2d}$$

$$k \frac{\partial \theta}{\partial x}(0, t) = -\frac{h_0}{t^{1/2}}, \quad t > 0 \tag{2e}$$

where $a^2 = k/\rho c$ represents the thermal diffusivity of the material and θ_0 is the temperature on the fixed face $x = 0$.

Remark 1

We suppose that $h_0 > 0$ and $\sigma > 0$ are known. The coefficient h_0 characterizes the heat flux on the fixed face $x = 0$, equation (2e), and σ characterizes the moving boundary (1). These must be determined experimentally. It is assumed that the temperature $\theta_0 > 0$ at $x = 0$ is given.

In Section 2, we shall consider six different cases with the following unknown coefficients:

$$\begin{aligned} & \text{(i) } k, c \quad \text{(ii) } k, l \quad \text{(iii) } k, \rho \\ & \text{(iv) } l, c \quad \text{(v) } l, \rho \quad \text{(vi) } c, \rho. \end{aligned} \tag{3}$$

We shall prove that there is not always a solution of the Lamé–Clapeyron–Neumann type [7, 18, 35, 45, 53, 54, 60] for problem (2) for all the six different cases (3). Namely, the explicit solution exists for cases (i), (iv), (v) and (vi) if and only if a complementary condition is satisfied; the explicit solution always exists for cases (ii) and (iii). This fact has been already observed in ref. [54] for the determination of an unknown coefficient of the phase change material and in refs. [52, 53] for other problems of the Stefan type.

The determination of the thermal conductivity $k = k(t)$ is studied in refs. [10, 27] through a one-phase Stefan problem. The determination of one or two unknown coefficients is studied in ref. [48] through a two-phase Stefan problem. A review of the control of parabolic systems involving free boundaries has been made [28] as has one on free boundary problems for the heat equation [44, 55]. The determination of thermal coefficients by other physical methods has been studied [1, 2, 4, 6, 25, 26, 32, 33, 39–41, 46, 51, 56, 57].

The main result of this paper can be stated as follows:

Main result

Knowing the three elements $\theta_0 > 0$, $h_0 > 0$ and $\sigma > 0$ given by equations (2d), (2e) and (1) respectively, it is possible to obtain formulas giving two of the four thermal coefficients k, l, c, ρ . Moreover, in cases (ii) and (iii) it is always possible to find an explicit solution to the associated inverse one-phase Lamé–Clapeyron

(Stefan) problem with the overspecified condition (2); in the remaining cases, the same conclusion is true if and only if a complementary condition is satisfied.

In Section 3, we shall consider the simultaneous determination of h_0 and one of the thermal coefficients k, l, ρ, c and we shall obtain the inequality (32) for the element ξ of the free boundary $s(t) = 2a\xi t^{1/2}$ of the Lamé–Clapeyron solution of the one-phase Stefan problem without unknown coefficients.

2. SOLUTION OF THE SIX DIFFERENT CASES

The solution of problem (2) is given by

$$\theta(x, t) = \theta_0 - \frac{\theta_0}{f(\sigma/a)} f(x/2at^{1/2}) \tag{4}$$

with

$$f(x) = \frac{2}{\pi^{1/2}} \int_0^x \exp(-u^2) du = \text{erf}(x)$$

where the two unknown coefficients, chosen between k, l, ρ, c , must satisfy the following system of equations:

$$\sigma \exp(\sigma^2/a^2) = h_0/\rho l \tag{5}$$

$$a/f(\sigma/a) = h_0\pi^{1/2}/\rho c\theta_0, \quad \text{with } a^2 = \frac{k}{\rho c}.$$

If we define

$$\xi = \frac{\sigma}{a} \tag{6}$$

we have

$$\exp(\xi^2) = h_0/\rho l\sigma, \tag{7a}$$

$$\xi f(\xi) = \rho c\sigma\theta_0/h_0\pi^{1/2}. \tag{7b}$$

Property 1. (Simultaneous determination of the coefficients k and c)

If the data $h_0 > 0$, $\sigma > 0$ and the coefficients of the phase change material $\rho > 0$, $l > 0$ verify the condition

$$\frac{h_0}{\rho l\sigma} > 1 \tag{8}$$

independently of θ_0/θ_0 , then problem (2) has the solution (4) where k and c are given by

$$c = \frac{h_0\pi^{1/2}}{\rho\sigma\theta_0} \xi f(\xi), \quad k = \frac{\sigma h_0\pi^{1/2} f(\xi)}{\theta_0 \xi} \tag{9}$$

and ξ is the unique solution of the equation

$$\exp(x^2) = \frac{h_0}{\rho l\sigma}, \quad x > 0 \tag{10a}$$

which is given by

$$\xi = \left[\log \left(\frac{h_0}{\rho l\sigma} \right) \right]^{1/2}. \tag{10b}$$

Property 2. (Simultaneous determination of the coefficients k and l)

For any data $h_0 > 0$, $\sigma > 0$, $\theta_0 > 0$ and for any

coefficients of the phase change material $\rho > 0, c > 0$, problem (2) has the solution (4) where k and l are given by

$$l = \frac{h_0}{\rho\sigma} \exp(-\xi^2), \quad k = \frac{\rho c \sigma^2}{\xi^2} \tag{11}$$

and ξ is the unique solution of the equation

$$xf(x) = \frac{\rho c \sigma \theta_0}{h_0 \pi^{1/2}}, \quad x > 0. \tag{12}$$

Property 3. (Simultaneous determination of the coefficients k and ρ)

For any data $h_0 > 0, \sigma > 0, \theta_0 > 0$ and for any coefficients of the phase change $l > 0, c > 0$, problem (2) has the solution (4) where k and ρ are given by

$$\begin{aligned} \rho &= \frac{h_0}{l\sigma} \exp(-\xi^2) = \frac{h_0 \pi^{1/2}}{c\sigma\theta_0} \xi f(\xi) \\ k &= \frac{h_0 c \sigma \exp(-\xi^2)}{l \xi^2} = \frac{\sigma h_0 \pi^{1/2}}{\theta_0} \frac{f(\xi)}{\xi} \end{aligned} \tag{13}$$

and ξ is the unique solution of the equation

$$xf(x) \exp(x^2) = \frac{c\theta_0}{l\pi^{1/2}}, \quad x > 0. \tag{14}$$

Property 4. (Simultaneous determination of the coefficients l and c)

If the data $h_0 > 0, \sigma > 0, \theta_0 > 0$ and the coefficient of the phase change material $k > 0$ verify the condition

$$\frac{k\theta_0}{2h_0\sigma} < 1 \tag{15}$$

independently of $\rho > 0$, then problem (2) has the solution (4) where l and c are given by

$$c = \frac{k}{\rho\sigma^2} \xi^2, \quad l = \frac{h_0}{\rho\sigma} \exp(-\xi^2) \tag{16}$$

and ξ is the unique solution of the equation

$$\frac{f(x)}{x} = \frac{k\theta_0}{h_0\sigma\pi^{1/2}}, \quad x > 0. \tag{17}$$

Property 5. (Simultaneous determination of the coefficients l and ρ)

If the data $h_0 > 0, \sigma > 0, \theta_0 > 0$ and the coefficient of the phase change material $k > 0$ verify the condition (15), independently of $c > 0$, then problem (2) has the solution (4) where l and ρ are given by

$$\rho = \frac{k}{c\sigma^2} \xi^2, \quad l = \frac{h_0 c \sigma \exp(-\xi^2)}{k \xi^2} \tag{18}$$

and ξ is the unique solution of equation (17).

Property 6. (Simultaneous determination of the coefficients c and ρ)

If the data $h_0 > 0, \sigma > 0, \theta_0 > 0$ and the coefficient of the phase change material $k > 0$ verify the condition (15), independently of $l > 0$, then problem (2) has the

solution (4) where c and ρ are given by

$$\rho = \frac{h_0}{l\sigma} \exp(-\xi^2), \quad c = \frac{kl}{h_0\sigma} \xi^2 \exp(\xi^2) \tag{19}$$

and ξ is the unique solution of equation (17). For the proofs of these properties see the Appendix.

Remark 2.

In cases (iv), (v) and (vi) the parameter ζ is defined by the same equation (17). Moreover, if we define c_{iv} and l_{iv} by equations (16), ρ_v and l_v by equations (18), and ρ_{vi} and c_{vi} by equations (19) corresponding to cases (iv), (v) and (vi) respectively, we have:

- (a) if we put $\rho = \rho_{vi}$ in the definition of c_{iv} , we obtain c_{vi} ;
- (b) if we put $\rho = \rho_v$ in the definition of l_{iv} , we obtain l_v ;
- (c) if we put $c = c_{vi}$ in the definition of ρ_v , we obtain ρ_{vi} ;
- (d) if we put $c = c_{iv}$ in the definition of l_v , we obtain l_{iv} ;
- (e) if we put $l = l_v$ in the definition of ρ_{vi} , we obtain ρ_v ;
- (f) if we put $l = l_{iv}$ in the definition of c_{vi} , we obtain c_{iv} .

Remark 3

The solution $\{\theta(x, t), s(t)\}$ of the Lamé-Clapeyron (Stefan) problem without unknown coefficients [7, 18, 35, 45, 54, 60], defined by (2a)-(2d) and $s(0) = 0$, is given by

$$\begin{aligned} \theta(x, t) &= \theta_0 - \frac{\theta_0}{f(\xi)} f(x/2at^{1/2}), \\ s(t) &= 2a\xi t^{1/2} \quad (\text{free boundary}) \end{aligned} \tag{20}$$

where ξ is the unique solution of equation (14). In this case, we have $h_0 = k\theta_0/af(\xi)\pi^{1/2}$, thus, the inequalities (8) and (15) are verified. The condition (15) is always verified because of the well-known properties of the error function, and condition (8) gives us the following inequality:

$$\xi f(\xi) < \frac{Ste}{\pi^{1/2}}, \tag{21}$$

$$Ste = \frac{c\theta_0}{l} \quad (\text{Stefan number})$$

which is of physical interest for all Stefan numbers and can be obtained trivially from equation (14).

Moreover, for the coefficient ζ , the following inequality is obtained in ref. [54, equation (29)]:

$$f(\xi) < \left(\frac{2}{\pi} Ste\right)^{1/2} \tag{22}$$

which is only of physical interest when $Ste < \pi/2$. In this case, we replace $<$ by $=$ and we denote by ξ_1 and ξ_2 the solutions of the equations from the inequalities (22) and (21), respectively. We get $\xi_2 < \xi_1$ since we have $f(x)$

$< (2/\pi^{1/2})x, \forall x > 0$, and $xf(x)$ is an increasing function in \mathbb{R}^+ .

Remark 4

In case (iii) of this work, and cases (i) and (iv) of ref. [54], the parameter ξ is defined by the same equation (14). Moreover, if we define ρ_{iii} and k_{iii} by equations (13), σ_1 and k_1 (see equation (11) of ref. [54]), and σ_4 and ρ_4 (see equation (25) of ref. [54]) by

$$\sigma_1 = \frac{h_0}{\rho l} \exp(-\xi^2), \quad k_1 = \frac{\pi h_0^2}{\rho c \theta_0^2} f^2(\xi), \quad (23)$$

$$\sigma_4 = \frac{k \theta_0}{h_0 \pi^{1/2}} \frac{\xi}{f(\xi)}, \quad \rho_4 = \frac{\pi h_0^2}{k c \theta_0^2} f^2(\xi), \quad (24)$$

corresponding to case (iii) of this work, and cases (i) and (iv) of ref. [54] respectively, we have:

- (a) if we put $\sigma = \sigma_4$ in the definition of ρ_{iii} , we obtain ρ_4 ;
- (b) if we put $\sigma = \sigma_1$ in the definition of k_{iii} , we obtain k_1 ;
- (c) if we put $\rho = \rho_{iii}$ in the definition of k_1 , we obtain k_{iii} ;
- (d) if we put $k = k_{iii}$ in the definition of ρ_4 , we obtain ρ_{iii} .

This remark completes ref. [54] Remark 1.

3. SIMULTANEOUS DETERMINATION OF OTHER PHYSICAL COEFFICIENTS

We shall consider the simultaneous determination of h_0 and one of the thermal coefficients k, l, ρ, c . Using a method similar to the one developed in Section 2, we obtain:

Property 7

(i) For any data $\sigma, \theta_0 > 0$ and for any coefficients of the phase change material $k, \rho, c > 0$, the simultaneous determination of $h_0, l > 0$ is given by

$$h_0 = \frac{k \theta_0}{\sigma \pi^{1/2}} \frac{\xi}{f(\xi)}, \quad l = \frac{c \theta_0}{\pi^{1/2}} \frac{\exp(-\xi^2)}{\xi f(\xi)} \quad (25)$$

with $\xi = \frac{\sigma}{a}$.

(ii) For any data $\sigma, \theta_0 > 0$ and for any coefficients of the phase change material $l, \rho, c > 0$, the simultaneous determination of $h_0, k > 0$ is given by

$$h_0 = \rho l \sigma \exp(\xi^2), \quad k = \frac{\rho c \sigma^2}{\xi^2} \quad (26)$$

where ξ is the solution of equation (14).

(iii) For any data $\sigma, \theta_0 > 0$ and for any coefficients of the phase change material $k, l, c > 0$, the simultaneous determination of $h_0, \rho > 0$ is given by

$$h_0 = \frac{k l}{c \sigma} \xi^2 \exp(\xi^2), \quad \rho = \frac{k}{c \sigma^2} \xi^2 \quad (27)$$

where ξ is the solution of equation (14).

(iv) If the data $\sigma, \theta_0 > 0$ and the coefficients of the phase change material $k, l, \rho > 0$ verify the condition

$$\frac{k \theta_0}{2 l \rho \sigma^2} > 1 \quad (28)$$

then the simultaneous determination of $h_0, c > 0$ is given by

$$h_0 = \rho l \sigma \exp(\xi^2), \quad c = \frac{k}{\rho \sigma^2} \xi^2 \quad (29)$$

where ξ is the solution of the following equation:

$$f(x) \exp(x^2) = \frac{k \theta_0}{l \rho \sigma^2 \pi^{1/2}} x, \quad x > 0. \quad (30)$$

Moreover, for the four cases the temperature θ [solution of problems (1) and (2)] is given by equation (4).

Remark 5

If $\sigma > 0$ is given, then the simultaneous determination of $\{\theta, h_0, j\}$ with $j \in \{k, l, \rho, c\}$, given by Property 7, is equivalent to the determination of $\{\theta, j\}$ [solution of problems (1), (2a)–(2d)] and h_0 given by

$$h_0 = \frac{k \theta_0}{a f(\sigma/a) \pi^{1/2}}. \quad (31)$$

Remark 6

From equation (28), we deduce the following inequality for the parameter ξ of the free boundary of the Lamé–Clapeyron solution of the one-phase Stefan problem without unknown coefficients (20) (cf. Remark 3)

$$\xi^2 < \frac{Ste}{2} \quad (32)$$

which is of physical interest for all Stefan numbers Ste .

If we denote by ξ_3 the solution of the equation obtained from the inequality (32) by replacing $<$ with $=$, i.e. $\xi_3 = (Ste/2)^{1/2}$, we get $\xi_3 < \xi_2$ (cf. Remark 3).

Remark 7. (Simultaneous determination of the coefficients k and α)

If the coefficient $\rho > 0$ is a data, then the simultaneous determination of k and α is equivalent to the simultaneous determination of k and c . Moreover, if the data verify the condition (8), then the coefficients k and α are given by

$$k = \frac{h_0 \sigma \pi^{1/2}}{\theta_0} \frac{f\{[\log(h_0/\rho l \sigma)]^{1/2}\}}{[\log(h_0/\rho l \sigma)]^{1/2}}, \quad (33)$$

$$\alpha = \frac{\sigma^2}{\log(h_0/\rho l \sigma)}.$$

Remark 8

If the coefficient $\rho > 0$ is a data, then the simultaneous determination of σ and α is equivalent to the simultaneous determination of σ and c (cf. [54]).

Moreover, if the data verify the condition

$$\frac{\rho k l \theta_0}{2h_0^2} < 1 \quad (34)$$

then σ and α are given by

$$\sigma = \frac{h_0}{\rho l} \exp(-\xi^2), \quad \alpha = \frac{h_0}{\rho^2 l^2} \frac{\exp(-2\xi^2)}{\xi^2} \quad (35)$$

where ξ is the solution of the equation

$$\frac{f(x)}{x} = \frac{\rho l k \theta_0}{h_0^2 \pi^{1/2}} \exp(x^2), \quad x > 0. \quad (36)$$

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APPENDIX

Proof of Property 3

By removing ρ in equation (7) we obtain for ζ the equation (14), the function

$$G_1(x) = xf(x) \exp(x^2), \quad (\text{A1})$$

defined for $x > 0$, has the following properties:

$$G_1(0^+) = 0, \quad G_1(+\infty) = +\infty, \quad G_1' > 0. \quad (\text{A2})$$

It follows from equations (A2) that equation (14) has a unique solution $\xi > 0$. The coefficient ρ is obtained from equations (7a) and (14), and the coefficient k is obtained from

$$a^2 = \frac{k}{\rho c} = \frac{\sigma^2}{\xi^2}, \quad (\text{A3})$$

using equation (14) and the value of ρ already calculated.

Proof of Property 4

From equation (A3), we have

$$c = \frac{k\xi^2}{\rho\sigma^2}. \quad (\text{A4})$$

From equations (A4) and (7b) we obtain equation (17) for ξ . The function

$$G_2(x) = \frac{f(x)}{x}, \quad (\text{A5})$$

defined for $x > 0$, has the following properties:

$$G_2(0^+) = \frac{2}{\pi^{1/2}}, \quad G_2(+\infty) = 0, \quad G_2' < 0. \quad (\text{A6})$$

It follows from equations (A6) that equation (17) has a unique solution if and only if the condition (15) is verified. Then the coefficient l is obtained from equation (7a).

To prove the remaining properties and remarks, we use a method similar to the one developed above and to that in ref. [54].

DETERMINATION SIMULTANEE DE DEUX COEFFICIENTS THERMIQUES INCONNUS
PAR UN PROBLEME INVERSE A UNE PHASE DE LAME-CLAPEYRON (STEFAN) AVEC
UNE CONDITION SUPERSPECIFIEE SUR LA FACE FIXE

Résumé—On donne des formules pour la détermination de deux des quatre coefficients k (conductivité thermique), l (chaleur latente de fusion), c (chaleur massique), ρ (masse volumique) d'un matériau semi-infini. Cette détermination est obtenue à travers un problème inverse monophasique de Lamé-Clapeyron (Stefan) avec une condition surspécifiée sur la face fixe du matériau qui change de phase. Pour résoudre ce problème, nous supposons que les coefficients h_0 , σ , $\theta_0 > 0$ sont connus par l'expérience (où h_0 caractérise le flux thermique à travers la face fixe, σ caractérise la frontière mobile et θ_0 est la température de la face fixe). Notant par θ la température, les résultats concernant le problème associé de frontière mobile est : (1) quand un des triplets (θ, k, l) , (θ, k, ρ) est à trouver le problème correspondant de frontière mobile a toujours une solution de type Lamé-Clapeyron-Neumann, (2) si un des triplets (θ, k, c) , (θ, l, c) , (θ, l, ρ) et (θ, c, ρ) est à déterminer, la propriété ci-dessus est satisfaite si et seulement si une condition complémentaire pour les données est vérifiée.

On obtient des formules pour la détermination simultanée des autres coefficients physiques et l'inégalité $\xi^2 < Ste/2$ (Ste , nombre de Stefan) pour le coefficient ξ de la frontière libre $s(t) = 2a\xi t^{1/2}$ de la solution de Lamé-Clapeyron du problème à une phase de Stefan sans coefficient inconnu.

SIMULTANE BESTIMMUNG ZWEIER UNBEKANNTER THERMISCHER KOEFFIZIENTEN
DURCH EIN INVERSES EINPHASIGES LAMÉ-CLAPEYRON-(STEFAN)-PROBLEM MIT
EINER ÜBERBESTIMMTEN BEDINGUNG AUF DER FESTEN SEITE

Zusammenfassung—Es werden Gleichungen für die simultane Bestimmung von zwei der vier Koeffizienten k (Wärmeleitfähigkeit), l (Schmelzwärme), c (spezifische Wärmekapazität), ρ (Dichte) eines halbunendlich ausgedehnten Körpers gefunden. Die Bestimmung wird durch ein inverses einphasiges Lamé-Clapeyron-(Stefan-)Problem mit einer überbestimmten Bedingung an der festen Seite des phasenwechselnden Mediums erhalten. Um dieses Problem zu lösen, nehmen wir an, daß die Koeffizienten h_0 , σ , $\theta_0 > 0$ aus Messungen bekannt sind, (wobei h_0 die Wärmestromdichte durch die feste Hälfte darstellt, σ die wandernde Grenzfläche und θ_0 die Temperatur der festen Begrenzungsfläche). Wenn man die Temperatur mit θ bezeichnet, so erhält man folgende Ergebnisse für das entsprechende Problem mit wandernder Grenzfläche:

(1) Wenn eines der Tripel $\{\theta, k, l\}$, $\{\theta, k, \rho\}$ gesucht ist, hat das zugehörige Problem mit wandernder Grenzfläche eine Lösung vom Lamé-Clapeyron-Neumann-Typ.

(2) Wenn eines der Tripel $\{\theta, k, c\}$, $\{\theta, l, c\}$, $\{\theta, l, \rho\}$ und $\{\theta, c, \rho\}$ zu bestimmen ist, wird die obige Eigenschaft erfüllt, wenn und nur wenn eine komplementäre Bedingung für die Daten verifiziert wird.

Weiterhin erhalten wir Gleichungen für die simultane Bestimmung anderer physikalischer Koeffizienten und die Ungleichung $\xi^2 < Ste/2$ (Ste : Stefan-Zahl) für den Koeffizienten ξ der freien Grenzfläche $s(t) = 2a\xi t^{1/2}$ der Lamé-Clapeyron-Lösung des einphasigen Stefan-Problems ohne unbekanntes Koeffizienten.

ОДНОВРЕМЕННОЕ ОПРЕДЕЛЕНИЕ ДВУХ НЕИЗВЕСТНЫХ КОЭФФИЦИЕНТОВ
ТЕПЛОПЕРЕНОСА ПУТЕМ РЕШЕНИЯ ОБРАТНОЙ ОДНОФАЗНОЙ ЗАДАЧИ
ЛАМЕ-КЛАПЕЙРОНА (СТЕФАНА) С ЗАДАННЫМ УСЛОВИЕМ НА
ФИКСИРОВАННОЙ ПОВЕРХНОСТИ

Аннотация—Получены выражения для одновременного определения каких-либо двух из следующих четырех коэффициентов материала, заполняющего полубесконечное пространство: k (коэффициент теплопроводности), l (скрытая теплота плавления), λ (теплоемкость), ρ (массовая плотность). Выражения получены из решения обратной однофазной задачи Ламе-Клапейрона (Стефана) с заданным условием на фиксированной поверхности материала при изменении его агрегатного состояния. При решении задачи предполагается, что коэффициенты h_0 , σ , $\theta_0 > 0$ известны из эксперимента (здесь h_0 — тепловой поток через фиксированную поверхность, σ характеризует подвижную границу, θ_0 — температура на фиксированной поверхности). Обозначив температуру через θ , из решения задачи с подвижной границей вытекает следующее: 1) в случае, когда необходимо определить один из наборов коэффициентов $\{\theta, k, l\}$ или $\{\theta, k, \rho\}$, задача всегда имеет решение в виде решения Ламе-Клапейрона-Неймана; 2) если определению подлежит один из наборов коэффициентов $\{\theta, k, \lambda\}$, $\{\theta, l, \lambda\}$, $\{\theta, l, \rho\}$ или $\{\theta, \lambda, \rho\}$, оговоренное выше решение возможно только в том случае, если выполнено условие привлечения дополнительных данных. Кроме того, получены выражения для одновременного определения других физических коэффициентов, а также коэффициента ξ из неравенства $\xi^2 < Ste/2$, характеризующего подвижную границу $s(t) = 2a\xi\sqrt{t}$ в решении Ламе-Клапейрона однофазной задачи Стефана при отсутствии неизвестных коэффициентов (Ste — число Стефана).